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Quantum mechanics in curved space-time

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Abstract. In this paper, the principles of the general relativity are used to formulate quantum wave equations for spin-0 and spin-1/2 particles. More specifically, the equations are worked in a Schwarzschild like metric. As a test, the hydrogen atom spectrum is calculated. A comparison of the calculated spectrum with the numerical data of the deuterium energy levels shows a significant improvement of the accord, and the deviations are almost five times smaller then the ones obtained with the Dirac theory. The implications of the theory considering the strong interactions are also discussed.

1 Introduction

The general theory of the relativity, proposed in 1916 by Einstein, was one of the major scientific discoveries of last century. Besides providing very accurate theoretical results, it was a great advance in the understanding the Nature, dealing with the structure of the space-time.

A question of interest is how the quantum theory can be affected by the space-time. Dirac, formulated his theory [1], based in the flat space-time of the special theory of the relativity, and with this formulation, the spin and the antiparticles appeared naturally into the theory.

About curved spaces, many authors, as for example [2,3], [4] proposed methods to quantize the gravity, where still there are many difficulties and opened questions to be understood.

In this work, a different point of view is proposed. Here, instead of trying to quantize the gravity, the effects of the metric in the subatomic world will be studied. For this purpose, the basic idea is to describe a particle in a region with a potential that affects the metric of the space-time. We are not interested in gravitational effects, as in [5], where the effect of gravitational forces in the hydrogen atom spectrum has been included. So, the gravitational potential will be turned off and only the other interactions (strong, electromagnetic) will be considered. Observing that the masses of the particles are very small, and the small value of the gravitational coupling, when compared with the electric or strong ones, one can say that it is an excellent approximation. Inside this space-time, curved by the interaction, according to the general covariance, quantum wave equations will be proposed. Then some simple applications will be made, in order to verify the predictions of the theory.

This paper will show the following contents: In Sect. 2 the operators in the Schwarzschild metric will be calculated, in Sect. 3, a brief review of the dynamics will be made, in Sects. 4 and 5 the quantum wave equations will be proposed. In Sect. 6 we will apply the theory to the hydrogen atom, calculating its energy spectrum and in Sect. 7, the strong interactions will be considered. In Sect. 8, the conclusions will be presented.

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2 The metric

In this section we will calculate the operators (E, $\boldsymbol{p}$, p^2) needed in order to write the wave equations, using the general relativity principles. As a first step, a system with spherical symmetry will be considered, but the basic ideas can be generalized to systems with arbitrary metrics.

We will consider a particle inside a field, that may be described by a potential function V. The source of the field (a mass for a gravitational field or a charge for an electromagnetic field) will have some distribution, described by a tensor $T_{\mu\nu} \neq 0$, in a certain space region.

Outside of the source distribution, on the empty space (where $T_{\mu\nu} \neq 0$) if we consider a system that presents spherical symmetry, with a central potential $V(r)$, the space-time may be described by a Schwarzschild like metric [6–8],

$$ds^2 = \xi \; d\tau^2 - r^2(d\theta^2 + \sin^2\theta \; d\phi^2) - \xi^{-1} dr^2 \quad , \tag{1}$$

where $\xi(r)$ is determined by the interaction potential $V(r)$, and is a function only of r, for a time independent interaction. $\xi(r)$ will be studied in detail in Sec. III. As we can see in (1), the metric tensor $g_{\mu\nu}$ is diagonal

$$g_{\mu\nu} = \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & -\xi^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \quad , \tag{2}$$

and can be written in the form

$$g^{\mu\nu} = h_\mu^{-2}\eta^{\mu\nu} \quad , \tag{3}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad . \tag{4}$$

Using these definitions, we can calculate the operators

$$\nabla_i = h_i^{-1}\frac{\partial}{\partial x^i} \quad , \tag{5}$$

$$\boldsymbol{p} = -i\hbar\boldsymbol{\nabla} \quad . \tag{6}$$

According to the above expressions, the momentum operator may be defined as

$$\boldsymbol{p} = -i\hbar\left[\hat{r}\sqrt{\xi}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\,\sin\,\theta}\frac{\partial}{\partial\phi}\right] \quad , \tag{7}$$

that in a region with $V = 0$ ($\xi = 1$) is the usual momentum operator in spherical coordinates

$$\boldsymbol{p} = -i\hbar\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\,\sin\,\theta}\frac{\partial}{\partial\phi}\right] \quad . \tag{8}$$

The energy operator is defined as

$$E = i\hbar\nabla_0 = \frac{i\hbar}{h_0}\frac{\partial}{\partial t} = \frac{i\hbar}{\sqrt{\xi}}\frac{\partial}{\partial t} \tag{9}$$

that for $V = 0$,

$$E = i\hbar\frac{\partial}{\partial t}. \tag{10}$$

The Laplacian is calculated using [7]

$$\nabla^2 = (h_1h_2h_3)^{-1}\left[\frac{\partial}{\partial x_1}\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{h_1h_3}{h_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\right] \quad , \tag{11}$$

where h_i are given in (2), so

$$|\boldsymbol{p}|^2 = -\hbar^2\left[\frac{\sqrt{\xi}}{r^2}\frac{\partial}{\partial r}\left(r^2\sqrt{\xi}\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\,\theta}\frac{\partial}{\partial\theta}\left(\sin\,\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad . \tag{12}$$

If one defines the momentum components as $p_i = -i\hbar\nabla_i$, one can observe that they are not good operators, these operators does not commute and are not even Hermitians, so the definition [10]

$$p_i = \frac{1}{\sqrt{D}}\frac{\partial}{\partial x^i}\sqrt{D} \quad , \tag{13}$$

where $D = \sqrt{-g}$, with $g = \det(g_{\mu\nu})$, will be used. With this definition,

$$p_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \tag{14}$$

$$p_\theta = -i\hbar\left(\frac{\partial}{\partial\theta} + \mathrm{cotg}\theta\right) \tag{15}$$

$$p_\phi = -i\hbar\left(\frac{\partial}{\partial\phi}\right) \quad , \tag{16}$$

and the commutation rules are

$$\left[p_i, q^j\right] = -i\hbar\delta_i^j \tag{17}$$

$$[p_i, p_j] = \left[q^i, q^j\right] = 0 \quad . \tag{18}$$

The Laplacian can be expressed as

$$\nabla^2 = \left(p_r - \frac{i\hbar}{\xi}\frac{\partial\xi}{\partial r}\right)\xi\left(p_r - \frac{i\hbar}{\xi}\frac{\partial\xi}{\partial r}\right) + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2 + \frac{3}{4}\frac{\partial\xi}{\partial r} - \frac{1}{4}\frac{\partial^2\xi}{\partial r^2} \quad , \tag{19}$$

that for weak potentials is just

$$\nabla^2 = p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2 \quad . \tag{20}$$

With the operators calculated in this section, one can obtain the relativistic quantum wave equations. If another symmetry is important (as axial symmetry, for example), the operators can be obtained in a similar way in the given metric.

3 Schwarzschild dynamics

In order to obtain the wave equations, two expressions are needed: the energy and $\xi(r)$. Then, also with the function of setting the notation, it is useful to make a brief review of the dynamics, in the Schwarzschild metric, and to show how the quantities of interest can be expressed.

From (1), the proper time is

$$d\tau_0 = \sqrt{ds^2} = d\tau\sqrt{\xi - \frac{\beta_r^2}{\xi} + r^2\beta_t^2} = d\tau/\gamma_s \quad , \tag{21}$$

with

$$\gamma_s = \frac{1}{\sqrt{\xi - \frac{\beta_r^2}{\xi} + r^2\beta_t^2}} \quad , \tag{22}$$

where β_r and β_t are the the radial and transverse parts of $\boldsymbol{\beta}$, respectively. They are defined as

$$\boldsymbol{\beta} = \frac{d\boldsymbol{x}}{d\tau} \quad , \tag{23}$$

$$\beta_r = \frac{dr}{d\tau} \quad , \tag{24}$$

$$\beta_t = \left[\left(\frac{d\theta}{d\tau}\right)^2 + \left(\frac{d\phi}{d\tau}\right)^2\sin^2\theta\right]^{1/2} \tag{25}$$

The principle of least action states that

$$S = \int L\ dt = -m_0 c^2 \int ds$$
$$= -m_0 c^2 \int d\tau \sqrt{\xi - \frac{\beta_r^2}{\xi} + r^2 \beta_t^2} \quad , \qquad (26)$$

where m_0 is the rest mass of the particle. Then, the Lagrangian can be expressed as

$$L = -m_0 c^2 \sqrt{\xi - \frac{\beta_r}{\xi} + r^2 \beta_t^2} = -m_0 c^2 / \gamma_s \quad . \qquad (27)$$

The momentum four-vector is defined as

$$p^\mu = E_0 \beta^\mu = E_0 \gamma_s (1, \boldsymbol{\beta}) = (p^0, \boldsymbol{p}) \qquad (28)$$
$$p_\mu = (\xi\ p^0, -\xi^{-1}\ p^1, -r^2 p^2, -r^2 \sin^2\theta\ p^3) \quad , \qquad (29)$$

where

$$E_0 = m_0 c^2 \quad . \qquad (30)$$

The equivalence principle provides the relation

$$\frac{d\beta^0}{d\tau} = -\Gamma^0_{\mu\nu} \beta^\mu \beta^\nu \quad , \qquad (31)$$

that gives

$$\gamma_s \frac{dp_0}{d\tau} = E_0 \Gamma^\sigma_{0\nu} \beta_\sigma \beta^\nu = E_0 \left[\Gamma^1_{00} \beta_1 \beta^0 + \Gamma^0_{01} \beta_0 \beta^1 \right] = 0 \quad . \qquad (32)$$

So, the energy defined as

$$p_0 = E = \xi \gamma_s E_0 = \frac{m_0 c^2 \xi}{\sqrt{\xi - \frac{\beta_r^2}{\xi} + r^2 \beta_t^2}} \quad , \qquad (33)$$

is a constant of motion. The other constant is $L_z = p_3/c$.

In the rest frame of the particle

$$p_{0\mu} p_0^\mu = -E_0^2 = -m_0^2\ c^4 \quad , \qquad (34)$$

that is a Lorentz invariant, so

$$p_\mu p^\mu = p^2 c^2 - \frac{E^2}{\xi} = -m_0^2\ c^4 \qquad (35)$$

and then, the energy relation is

$$\frac{E^2}{\xi} = p^2 c^2 + m_0^2 c^4 \quad , \qquad (36)$$

or

$$\frac{E}{\sqrt{\xi}} = \sqrt{p^2 c^2 + m_0^2 c^4} \quad . \qquad (37)$$

The expressions (36) and (37) will be used to construct the Hamiltonian operators.

The term ξ is a function of r, and can be determined if we observe (37)

$$E(\beta = 0) = E_0 \xi^{1/2} = E_0 + V \quad , \qquad (38)$$

that means that in the rest frame of the particle, the energy is due to the sum of its rest mass with the potential. Then

$$\xi^{1/2} = 1 + \frac{V}{E_0} = 1 + \frac{V}{m_0 c^2} \quad . \qquad (39)$$

Comparing with the standard definition of the Schwarzschild mass

$$\xi = 1 - \frac{2\ m_s}{r} = 1 + \frac{2V}{m_0 c^2} + \frac{V^2}{m_0^2 c^4} \quad , \qquad (40)$$

it is possible to make the identification

$$m_s = -\frac{r}{2} \left(\frac{2V}{m_0 c^2} + \frac{V^2}{m_0^2 c^4} \right) \quad , \qquad (41)$$

that for general potentials may be a function of r.

For weak potentials,

$$\xi \sim 1 + \frac{2V}{m_0 c^2} \quad . \qquad (42)$$

and

$$m_s \sim -\frac{Vr}{m_0 c^2} \qquad (43)$$

that are the usual expressions of general relativity,

$$\xi_G = \left(1 - \frac{GM}{r\ c^2} \right)^2 \sim 1 - \frac{2GM}{r\ c^2} \quad . \qquad (44)$$

4 Spin-0 particles wave equation

With the knowledge of the energy equation (36) and $\xi(r)$ (39), it is possible to formulate wave equations in the Schwarzschild metric. The simplest case is to obtain the equation for spin-0 particles. For this purpose, the procedure to be followed is the same one that is used to determine the Klein-Gordon equation, that is based in an operator for E^2. Using the relation (9)

$$\frac{E^2}{\xi} = -\frac{\hbar^2}{\xi} \frac{\partial^2}{\partial t^2} \qquad (45)$$

and (36) the quantum wave equation, based on general relativity, for spin-0 particles is

$$-\frac{\hbar^2}{\xi^2} \frac{\partial^2 \Psi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \Psi + m_0^2 c^4 \Psi \quad , \qquad (46)$$

with ∇^2 defined in (11).

This equation can be separated in the standard way, yielding

$$\Psi(r, \theta, \phi, t) = u(r) Y_l^m(\theta, \phi) \mathrm{e}^{-iEt/\hbar} \quad , \qquad (47)$$

where $Y_l^m(\theta, \phi)$ are the spherical harmonics. The radial equation is then

$$\frac{\sqrt{\xi}}{r^2} \frac{\partial}{\partial r} \left(r^2 \sqrt{\xi} \frac{\partial}{\partial r} \right) u + \left(\frac{E^2}{\hbar^2 c^2 \xi^2} - \frac{m_0^2 c^2}{\hbar^2} - \frac{l(l+1)}{r^2} \right)\ u = 0 \ , \qquad (48)$$

and can be solved for a given interaction potential $V(r)$, that determines $\xi(r)$.

5 Spin-1/2 particles wave equations

The next step, is to obtain the analog of the Dirac equation, for spin-1/2 particles. It can be made in the same way that it was made by Dirac [1], using an Hamiltonian with the $\boldsymbol{\alpha}$ and β Dirac matrices, instead of an energy operator with an square root (37). Then we have

$$\frac{i\hbar}{\xi}\frac{\partial}{\partial t}\Psi = \left(-i\hbar c\, \boldsymbol{\alpha}.\boldsymbol{\nabla} + \beta m_0 c^2\right)\Psi \quad , \tag{49}$$

and if we square the operators in both sides of the equation, we obtain the wave equation (46), that proofs that the procedure used by Dirac, is also valid in this case.

Separating the time dependent part

$$T(t) = A\ \mathrm{e}^{-iEt/\hbar} \quad , \tag{50}$$

we will have the spatial equation,

$$\left(-i\hbar c\, \boldsymbol{\alpha}.\boldsymbol{\nabla} + \beta m_0 c^2 - \frac{E}{\sqrt{\xi}}\right)\psi(\boldsymbol{r}) = 0 \quad . \tag{51}$$

Observing the relation

$$\boldsymbol{\alpha}.\boldsymbol{\nabla} = \frac{\xi}{r}\boldsymbol{\alpha}.\boldsymbol{r}\left[\xi^{1/2} r\frac{\partial}{\partial r} + \alpha^r\left(\frac{\alpha^\theta}{r}\frac{\partial}{\partial\theta} + \frac{\alpha^\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right] \quad . \tag{52}$$

and the angular part of the operator of (52), one concludes that the angular part of $\psi(\boldsymbol{r})$ can be described in terms of the two component spinors χ_k^μ, [9],

$$\chi_k^\mu = \sum_{m=\pm 1/2} C(l,1/2,j;\mu-m,m)Y_l^{\mu-m}(\theta,\phi)\chi^m \quad , \tag{53}$$

where $C(l,1/2,j;\mu-m,m)$ is a Clebsh-Gordan coefficient, χ^m, a Pauli spinor and

$$\begin{aligned} k &= l && \text{for} \quad j = l - 1/2 \quad , \\ k &= -l-1 && \text{for} \quad j = l + 1/2 \quad , \end{aligned} \tag{54}$$

that gives

$$k = \pm(j + 1/2) \quad . \tag{55}$$

Then, the wave function is expected to have the structure

$$\psi = \begin{pmatrix} F(r)\chi_k^\mu \\ iG(r)\chi_{-k}^\mu \end{pmatrix} \quad , \tag{56}$$

with the F and G functions obeying

$$\begin{aligned} \sqrt{\xi}\frac{dF}{dr} + (1+k)\frac{F}{r} &= \left(\frac{E}{\sqrt{\xi}} + m_0\right)G \\ \sqrt{\xi}\frac{dG}{dr} + (1-k)\frac{G}{r} &= -\left(\frac{E}{\sqrt{\xi}} - m_0\right)F \quad , \end{aligned} \tag{57}$$

that are equations very similar to the ones obtained from the Dirac theory. In the following sections, some physical implications of this theory will be studied.

6 The hydrogen atom

In this section we will study the behavior of the theory, in a very well known system, the hydrogen atom. In the hydrogen atom, the electron is submitted to an electric central potential (obviously with $Z = 1$)

$$V(r) = -\frac{\alpha Z}{r} \quad , \tag{58}$$

then the ξ function becomes (where ep means electron-proton)

$$\xi_{ep} = \left(1 - \frac{\alpha Z}{m_e c^2\ r}\right)^2 \quad , \tag{59}$$

where m_e is the electron mass. This function represents a space-time curved by the ep interaction, or how space-time is seen by the electron. At this point we can see an interesting feature of the theory: when general relativity is used to study the gravitation, the radius where the metric breaks ($\xi = 0$) , that is the Schwarzchild radius, r_s, is always negligible, as for example r =2.95 Km for the sun. But in the hydrogen atom (with the parameters of [15])

$$r_s = \frac{\alpha}{m_e c^2} = 2.818 \text{ fm} \quad , \tag{60}$$

that is the classical radius of the electron, which is obviously not negligible. The estimated radius of the proton is about 0.9-1.0 fm, so, it is located inside the horizon of events. Then, the electric charge will be confined inside this region by a trapping surface, as defined in [14], and outside, only effects of the total charge can be probed by the electron, and no information about the inner structure can be obtained.

Inserting ξ_{ep} from the expression (59) in (57) we obtain the equations, but valid only for $r > r_s$. Inside the horizon of events, the metric is not the same, the energy-momentum tensor $T_{\mu\nu}$ determined by the charge and matter distributions must be considered. In this paper we shall study only the outside behavior. It is illustrative to study the approximation

$$\begin{aligned} \frac{dF}{dr} + (1+k)\frac{F}{r} &= \left(\frac{E}{1+V/E_0} + m_e\right)G \\ &\sim \left(E - \frac{EV}{E_0} + m_e\right)G \quad , \\ \frac{dG}{dr} + (1-k)\frac{G}{r} &= -\left(\frac{E}{1+V/E_0} - m_e\right)F \\ &\sim -\left(E - \frac{EV}{E_0} - m_e\right)F \quad , \end{aligned} \tag{61}$$

where we neglected the V/E_0 and higher order terms. As we can see, in this theory, the Dirac theory is recovered only for $E/E_0 \sim 1$ (lower moments),

$$\begin{aligned} \frac{dF}{dr} + (1+k)\frac{F}{r} &= (E - V + m_e)\,G \\ \frac{dG}{dr} + (1-k)\frac{G}{r} &= -\,(E - V - m_e)\,F \quad . \end{aligned} \tag{62}$$

We must remark that in the approximation of (61), the metric divergence at $r = r_s$ (that is not a physical singularity, it appears from the choice of the coordinate system) is removed, then $\xi \sim 1$ and r_s does not exist. So, in this case, the wave functions must be controlled only at the physical singularity, at the origin. Considering the Frobenius method, the solutions are of the type

$$F = \rho^s \sum_{n=0}^{N} a_n \rho^n \mathrm{e}^{-\rho} ,$$
$$G = \rho^s \sum_{n=0}^{N} b_n \rho^n \mathrm{e}^{-\rho} . \tag{63}$$

where $\rho = \beta r$, and after some manipulations one finds

$$\beta = \sqrt{m^2 c^4 - E^2} \tag{64}$$

and

$$s = \sqrt{k^2 - \frac{\gamma^2 E^2}{m^2}} . \tag{65}$$

Observing the solutions (63) one can see that $F(r \sim 0)$ and $G(r \sim 0)$ are sums of terms of the type $C\mathrm{e}^{-\rho}\rho^{s+m}(r \sim 0)$, and then, $\psi(r=0)$=0. So, the effect of the approximation, is to remove the horizon of events, and to extend the solution to the region $r < r_s$. Considering the exact solution of the equation, this behavior is valid only for $r > r_s$, and an horizon of events exists at this surface with the proprieties described above. But one must remark that at the atomic level, an approximation of the order of 2.8 fm is not so bad, in this region the wave functions are almost negligible and for practical purposes this approximation is reasonable.

From these expressions, it is possible to calculate the energy spectrum. Using the standard methods [9,11,12] one obtains

$$\left(m_\mathrm{e}^2 c^4 - E^2\right) a^2 = \frac{E^4 \gamma^2}{m_\mathrm{e}^2 c^4} , \tag{66}$$

that has the solutions

$$E^2 = m_\mathrm{e}^2 c^4 \left[\frac{-1 \pm \sqrt{1 + 4\gamma^2/a^2}}{2\,\gamma^2} \right] , \tag{67}$$

with

$$\gamma = \frac{\alpha}{\hbar c} , \tag{68}$$

and

$$a = a(n) = n - (j + 1/2) + \sqrt{(j+1/2)^2 - \gamma^2} . \tag{69}$$

Using only the positive root solutions of (67) the hydrogen atom spectrum is

$$E_n = m_e c^2 \sqrt{\frac{2}{1 + \sqrt{1 + 4\gamma^2/a^2}}} . \tag{70}$$

Adopting the expansion (γ^2/a^2 is small)

$$\sqrt{1 + 4\frac{\gamma^2}{a^2}} \sim 1 + 2\frac{\gamma^2}{a^2} - 2\frac{\gamma^4}{a^4} + 4\frac{\gamma^6}{a^6} - 10\frac{\gamma^8}{a^8} + 28\frac{\gamma^{10}}{a^{10}} + ... \tag{71}$$

the energy spectrum (70) can be rewritten as

$$E_n = \frac{m_e c^2}{\sqrt{1 + \gamma^2/a^2 - \gamma^4/a^4 + 2\gamma^6/a^6 - 5\gamma^8/a^8 + ...}} , \tag{72}$$

where we can find explicitly the corrections of the energy levels, due to general relativistic effects, if compared with the standard [11,12] relativistic spectrum

$$E_n = \frac{m_e c^2}{\sqrt{1 + \gamma^2/a^2}} , \tag{73}$$

that can be obtained from the Dirac equation or from the Sommerfeld model [13].

Considering now the spectrum obtained from the exact solution of (57), without the approximations made in (61)

$$E_N = m_e c^2 \sqrt{\frac{1}{2} - \frac{N^2}{8\alpha^2} + \frac{N}{4\alpha}\sqrt{\frac{N^2}{4\alpha^2} + 2}} , \tag{74}$$

we will compare the theoretical results with the experimental data and with the ones obtained with the Dirac theory.

Some numerical values are shown in Table I, the experimental results [16] for the differences between the energies $E(n,l,j)$ and the ground-state energies E_1, for the hydrogen atom and deuterium, the corresponding values calculated with the Dirac theory (73), and the results calculated in this work, with (74). Observing the table, one can see that the accord of both theories with the hydrogen experimental data is very good, but the results from (74) are closer to the experimental data then the results from (73). One must remark that spherical symmetry is not exact in the hydrogen atom as the proton mass is finite, but with a heavier nuclei, this symmetry is a better approximation, so it is interesting to observe the deuterium data. Comparing the results form the Dirac theory (73), one can see a better accord, and the deviations from the data are of the order of 0.027%. Considering the spectrum (74), the deviations are of the order of 0.005%, approximately five times smaller.

7 Strong interactions

In this section, the implications of the theory, when strong interactions are taken into account will be studied. The simplest system possible is the NN interaction. If, as a first approximation, only the long range part of the potential would be considered, it should be dominated by the one pion exchange contribution (Yukawa potential),

$$V(r) = g^2 \frac{\mathrm{e}^{-\mu r}}{r} \tag{75}$$

Table 1. Experimental energy levels (eV) for the hydrogen atom, for the deuterium [16], and the theoretical ones, calculated with the Dirac theory (73) and with (74)

	Hydrogen	Deuterium	Dirac	Eq. (74)
E_1	−13.59844	−13.60214	−13.60587	−13.60298
$E(2,0,1/2)-E_1$	10.19881	10.20159	10.20444	10.20172
$E(3,0,1/2)-E_1$	12.08750	12.09079	12.09413	12.09127
$E(4,0,1/2)-E_1$	12.74854	12.75201	12.75551	12.75263
$E(5,0,1/2)-E_1$	13.05450	13.05806	13.06164	13.05875

where $g^2 = 13.40$ is the NN coupling constant and μ is the pion mass. As $V(r)$ is a function only of r, if we locate one nucleon at the origin, we would have

$$\xi_{NN} = \left(1 - g^2 \frac{\mathrm{e}^{-\mu r}}{m_N c^2\, r}\right)^2 \quad . \tag{76}$$

However, as it is well known, the NN potential is not central, and there are other contributions, such as the tensor part [17–19] that arises from more complex processes (two pion exchange [20] and others). In order to make some estimates, some symmetrical cases of the Reid [18] potentials can be used, as they are phenomenological ones and can give a first idea of the Schwarzschild radius. The potentials are superpositions of Yukawa type terms,

$$V(^1S) = -h\left(\mathrm{e}^{-x} + 39.633\mathrm{e}^{-3x}\right)/x \quad , \tag{77}$$
$$V(^1D) = -h\left(\mathrm{e}^{-x} + 4.939\mathrm{e}^{-2x} + 154.7\mathrm{e}^{-6x}\right)/x \quad , \tag{78}$$
$$V(^1S)_s = -\left(h\mathrm{e}^{-x} + 1650.6\mathrm{e}^{-4x} - 6484.2\mathrm{e}^{-7x}\right)/x \quad , \tag{79}$$

where $x = 0.7\ r\ \mathrm{fm}^{-1}$, r is the relative radius and h=10.463 MeV. Inserting these potentials in (39) and equating it to 0, we will have

$$r_s(^1S) = 0.33\ \mathrm{fm} \quad , \tag{80}$$
$$r_s(^1D) = 0.44\ \mathrm{fm} \quad , \tag{81}$$
$$r_s(^1S)_s \sim 0.33\ \mathrm{fm} \quad . \tag{82}$$

Thus, the part of the source of the strong force that is inside the horizon of events will be submitted to the the trapping effect [14], that prevents the escape of any matter and radiation, what means radial collapse of the source of the strong forces.

It must be noted that the potentials containing spin dependent terms and others were not considered, fact that would break the spherical symmetry. However, in a first approximation, these terms can be considered as corrections to the central potential. But even if we consider these terms, with another metric, as for example an axial symmetric metric, the trapping surface would still exists, confirming the present conclusions, and, only giving a more accurate estimate of the size of the confining region. Collapse is not an exclusive feature of spherical symmetric systems, as it was stated in [14], deviations from spherical symmetry cannot prevent space-time singularities from arising.

Table 2. Values of the masses M compared with the experimental ones [15] for some systems. The calculations are made considering Coulomb potentials with the parameters α, m and r_s

	m (GeV)	α	r_s (fm)	M (GeV)	M_{exp} (GeV)
Nucleon(qqq)	0.38	1.60	0.83	0.938	0.938 (proton)
$J/\psi(c\bar{c})$	1.79	1.00	0.11	3.10	3.10
$\Upsilon(b\bar{b})$	5.50	1.05	0.05	9.47	9.46

As we can see, when strong interactions are considered, the horizon of events is located in a radius that is not negligible. The preceding example gives an estimate in the range of 0.3-0.5 fm. This fact suggests that the quark confinement may be understood from these results. To understand the mechanism, let us consider that the source of the strong force obeys some matter distribution. Each element of this matter distribution, (that may be a quark, but in general, this assumption is not necessary) with mass m_0, suffers the action of an attractive strong force. Some models [21–24] consider central Coulombic potentials of the type (58) to describe the effective interaction to which the quarks are submitted inside an hadron. A good example is the Cornell model [25], that with a linear plus Coulomb central potential

$$V(r) = -\frac{\alpha}{r} + \frac{r}{a^2} \quad , \tag{83}$$

with the parameters $a \sim 2.34\ \mathrm{GeV}^{-1}$ and $\alpha \sim 0.5$, is able to describe the J/ψ and the Υ.

Thinking in terms of constituent quarks it is possible to use the proposed theory to give a description of some hadrons. Table II shows an estimate of α (for a Coulomb potential) and of the constituent quark masses (m) in order to obtain the masses (M) of the nucleon and the J/ψ and Υ mesons. The experimental values of these masses are also shown. These constants define the value of r_s, inside of which the quarks are expected to be confined. The values of $r_s = 0.83$ fm for a nucleon and $0.05 - 0.11$ fm for heavy mesons are very reasonable and show that for heavier quarks, the values of r_s are smaller. Table II was constructed with the objective of giving an idea of the magnitude of the constants, but a detailed description of the observed hadrons must consider additional terms in the potential.

With these results, it is possible to calculate [26,27]

$$\frac{g_A}{g_V} = \frac{5}{3}\langle\sigma_Z\rangle = \frac{5}{3}(1 - 2\ \delta)\ , \tag{84}$$

where

$$\delta = \frac{\frac{2}{3}\int |G(r)^2| dr}{\int \left(|F(r)|^2 + |G(r)|^2\right) dr} = 0.059 \tag{85}$$

where a nucleon composed of tree quarks with $j_z = 1/2$ is considered. So, g_A/g_V=1.47, what shows a 17% deviation from the experimental result that is 1.259. The magnetic moments of the proton and the neutron may also be calculated

$$\mu_p = \frac{(1-\delta)m_p}{E_0} = 2.82$$
$$\mu_n = -\frac{2}{3}\frac{(1-\delta)m_p}{E_0} = -1.88 \ , \qquad (86)$$

that are in good agreement with the experimental results μ_p=2.79 and μ_n=-1.91 [15].

If the whole quark content of the hadron is located inside r_s (now, spherical symmetry is a good choice), the classical description of such a system would predict the collapse of the whole matter in the singularity located at $r = 0$. However, as we are dealing with a quantum system, the uncertainty principle will prevent this collapse. Consequently, there are two opposite effects acting on the elements of matter, and the resulting effect will be radial oscillations. This effect, in a flat Minkowski space-time may be described by effective potentials of the form

$$V_{\rm eff} \sim a_0 + a_1 r + a_2 r^2 + ... \ . \qquad (87)$$

There is no surprise why some authors [28,29], explain the hadronic structure with models based on harmonic oscillator quark models or with linear potentials of the type λr [30,25]. Inside the horizon of events, $V > E_0$, so (57) can be expanded as

$$\left(1-\frac{\alpha}{m_0 r}\right)\frac{dF}{dr} + (1+k)\frac{F}{r}$$
$$\sim \left\{-E\left[\frac{m_0 r}{\alpha} + \left(\frac{m_0 r}{\alpha}\right)^2 - \left(\frac{m_0 r}{\alpha}\right)^3 + ...\right] + m_0\right\} G$$
$$\left(1-\frac{\alpha}{m_0 r}\right)\frac{dG}{dr} + (1-k)\frac{G}{r}$$
$$\sim -\left\{-E\left[\frac{m_0 r}{\alpha} + \left(\frac{m_0 r}{\alpha}\right)^2 - \left(\frac{m_0 r}{\alpha}\right)^3 + ...\right] - m_0\right\} F \ , \qquad (88)$$

where terms similar to the effective potential (87) appears. So, considering a Coulombic potential (fact that is not strictly necessary, other kind of potentials may present similar proprieties) with the correct parameters, confinement effects may occur, fact that is widely used, with the addition of confining potentials in flat space-times.

8 Summary and conclusions

In this paper, quantum wave equations, based on the general relativity, in the Schwarzschild metric, have been obtained. Investigating the hydrogen atom spectrum, the approximate expression resulting from the theory (70) is in accord with the experimental values, and shows a small improvement due to the general relativistic corrections, when compared with the standard relativistic spectrum (73). Although, if the exact solution is considered, the corrections are not so small and (74) gives a significant improvement of the accord with the experimental data, specially with the deuteruim spectrum.

An interesting feature of the theory, is that in the Schwarzschild metric, the horizon of events appears for $r = r_s$, with a value that is not negligible, as it happens when the gravitational interaction is considered. When considering the strong interaction, r_s shows a region inside the hadron, where confinement arises. From this theory, confinement may be considered as an intrinsic propriety of the space-time, that when interactions with large coupling constants are considered, generates trapping surfaces. On the other hand, no collapse for $r = 0$ is expected, the uncertainty principle forbids it. ξ as defined in (39) is a function of α/m, so, in Nature, α and m are such that the confinement conditions are filled and in a region with the observed size (some examples are shown in Table II).

Thinking matter as composed of small black holes may seem a weird idea, but no one has actually seen a quark, or an element of strongly interacting matter, and in this sense, a black hole is quite reasonable. At astronomical level also, it is possible to imagine systems that cannot be seen, due to the curvature caused by electromagnetic or strong forces, and maybe giving an important contribution to the mass of the universe.

The most important feature of the theory, is the fact that the insertion of general relativistic aspects in the quantum theory generates results that are in accord with the phenomenology of the considered systems.

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